# Introduction

Sometimes it happens in mathematics that, investigating some branch, some area of mathematical objects, we are forced to introduce new kind of objects which were not originally presumed to exist. There is a classical example, well known to students of mathematical departments. We have in mind that it is hardly possible to develop some branches of real analysis without involving complex numbers. Another example concerns infinitesimal (or infinitely small) and infinitely large numbers.

During the early history of modern mathematics, from Kepler and Cavalieri (and, in some respects, from the great Archimedes) to Euler and Cauchy, infinitesimals were regarded as an integral part of Analysis. However the level of logic at that time did not allow the mathematics to define an exact, mathematically rigorous system of foundations of Infinitesimal Calculus. This problem remained unsolved until the early 1960's, when A.Robinson and his followers introduced and began to develop a new precise construction of infinitesimals, which is known now as *nonstandard analysis.*

This short book on nonstandard analysis is based upon two courses given by the author in Tehran (May – June 1992) and Wuppertal (April – June 1993), a lecture given in Pasadena (December 1992), and discussions made during 1992–1993.

Some rather popular topics (*superstructures, Loeb measures*) are considered in the first two chapters. Chapter 1 contains a nonformal introduction to nonstandard real line in Section 1. This is followed by "nonstandard" proofs of several principal theorems in foundations of analysis in Section 2, made in the form of a *simple logical transformation* of sentences. Superstructures, the most successful technical tool of nonstandard analysis, together with some related topics, are considered in Chapter 2.

The remainder of the book is devoted to set theoretic foundations of nonstandard mathematics, especially theories of internal sets.

Such a theory, *internal set theory,* IST, introduced by Nelson in 1977, has been endorsed by many experts as a suitable tool for develop in gvarious branches of nonstandard analysis from a common standpoint. The exposition of IST in Chapter 3 is made so that we pay attention mostly to several basic principles introduced by Nelson which govern the interactions of sets in IST universe: *external induction, extension, uniqueness, collection, reduction to* $\Sigma_2^{st}$ *form*. We prove in particular that the Uniqueness-and Collection theorems hold in IST for *all* formulas, including those containing the predicate of standardness (Section 9).

Chapter 4 develops some metamathematical questions related to IST in its connection to ZFC. Together with the classical *consistency* and *conservation* theorems of Nelson, we prove (Section 14) a principal *non-reducibility* result: there exists an explicitly defined sentence in the IST language not equivalent in IST to a sentence in the ZFC language. (This is discussed in detail in M.Reeken's preface.) This is based on the *adequate ultrapower* construction of IST enlargements of ZFC models (Section 12) and a modification of this method known as *definable ultrapower* (Section 13).

Another application of this technique is related to *independence proofs*: we prove that some forms of Extension, a nonstandard tool widely used in applications, are in fact independent of IST. Besides ultrapowers, these proofs involve a theorem of Section 11 which gives a common *truth definition* in IST for all ZFC formulas with standard parameters.

The final chapter presents BST, *bounded set theory*, a modification of IST, actually a theory which describes the class of those sets which are elements of standard sets in IST universe. This theory differs from IST is several remarkable details, including the following:

- Principal theorems: Extension and Reduction to $\Sigma_2^{st}$ theorems are provable in BST (unlike IST).

- Reducibility: every sentence is equivalent in BST to a sentence in the ZFC language (see discussion of this topic in the Preface).

- External sets. Such important objects as the "set" of all standard natural numbers – precisely those which need an appeal to the standardness predicate to be defined (we call them *classes*) – are *not* sets in IST. This problem can be fixed in IST in some cases (not in the case of Loeb measure) by *parametrization* of all those definable classes which actually participate in the reasoning. We prove (sections 16 and 17) that BST admits a parametrization of the family of all definable *bounded* (that is, subclasses of sets) classes by a certain explicitly given formula, which solves the problem of external sets in BST at the principal level.

It may be expected that BST will find interesting applications as a base for nonstandard mathematics.

In general the book is addressed to those doing applications of nonstandard methods in various fields of mathematics who want to obtain a new experience in foundations, as well as to logicians who do not mind getting acquainted with the foundations of nonstandard mathematics.

The exposition is more or less self-contained, although certain knowledge in set theoretic foundations of mathematics is assumed. A number of exercises, some of them rather difficult, are included with the aim of making the book useful as a textbook for students at graduate and advanced undergraduate level.

**Acknowledgements.** The author is in debt to many people for fruitful discussions during the preparation of the text.

Special thanks to Prof.Dr.M.Reeken for his taking a lot of time to discuss the content of the paper, outlining the principal ways of exposition, looking through the text, fixing a number of details, and writing the preface.

Separate thanks to M. J. A. Larijani, the President of the IPM (Tehran), A. Kechris (Caltech, Pasadena), S. Albeverio (Bochum), M. Reeken (Wuppertal) for their assistance in writing and publishing the book, and to A. Enayat, G. B. Khosrovshahi, W. A. J. Luxemburg, S. Shahshahani, Tom Linton, K. Maleki, N. Prinz, K.-G. Schlesinger, S. Vaheed, B. Wietschorke, as well as to Ms. A. Arfaee, Marge D'Elia, Frau Ackermann, and Herr Rasch, who made my stay in Tehran, Pasadena, and Wuppertal both successful and comfortable.

Wuppertal, June 1993
Tehran, December 1993 — Vladimir Kanovei