Acknowledgements

First of all I would like to thank Dr. G. B. Khosrovshahi for providing the opportunity of delivering a series of lectures at IPM and considering these notes for publication in IPM lecture notes series.

I would like to thank Dr. M. Shahshahani for inviting me to present the lectures in his number theory seminar which provided a great audience for my lectures. I also would like to thank the students participating in the lectures.

Mr. Mehdi Hassani carefully proof read these notes, I would like to thank him for his many valuable comments.

Parts of the notes of the third lecture is from Mr. Shahab Shahabi's M.Sc. thesis. I would like to thank him for allowing me to include them here.

Preface

In May 2006, I gave four lectures on classical analytic theory of L-functions at IPM. The following pages are notes that I prepared for those lectures. As you will see these notes are brief and do not follow a textbook style treatment of the subject. My goal was to give a quick introduction to L-functions, by selecting some fundamental topics from the classical theory. The lectures are interrelated and should be read in progression. Many statements are given as exercises. These notes should be read in a slow pace and I encourage the reader to do the exercises. I hope that these notes serve as a guideline for the beginners interested in the theory of L-functions.

Let me give a brief overview of the contents. In the first lecture we set up our notation and terminology. Here we consider L-functions as complex functions that satisfy some nice analytic properties. Following Iwaniec and Kowalski, we axiomatize a class of L-functions, which is basically a class of complex functions satisfying properties similar to automorphic L-functions. Next we introduce the Rankin-Selberg convolution of two L-functions in this class. The fundamental role of these convolutions in the theory of L-functions and their many applications form the main theme of these lectures. We illustrate the importance of these convolutions by describing their relations with the problem of finding sharp estimates for some arithmetic functions. Moreover, in the second lecture we show that how the existence of these convolutions will guarantee the non-vanishing of L-functions on the line $\Re(s) = 1$, and consequently will lead to the prime number theorem type results. In the third lecture we show that for two L-functions associated to cusps forms the Rankin-Selberg convolution exists, and as a consequence of this fact we can apply the results of Lecture 2 to deduce, in the fourth lecture, the prime number theorem type estimates for the Fourier coefficients of a cusp form.

Lectures 1 and 2 are based on chapter 5 of [IK][1], [R1], [O] and [GHL]. The third lecture gives a detailed exposition of the classical paper of Rankin [R2]. The final lecture is based on [Mo], [HL] and [R2].

[1] See the list of references at the end of notes.

We are using without proof many facts from complex analysis and Fourier analysis. [T] is a good complex analysis reference. [D], [I], [IK], and [M] are good analytic number theory references. For the basic material on modular forms the reader can consult [B], [CKM], [Iw], [K], [S], and [Sh].

[GM], [IS] and [Mi] are good survey articles and they include extensive bibliography.

I hope that these notes give a glimpse of this fascinating subject and motivate the reader for further studies of L-functions.

Amir Akbary
October 2006

Contents